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Causal theory for the gauged Thirring model

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Abstract. We consider the $(2+1)$-dimensional massive Thirring model as a gauge theory, with one-fermion flavor, in the framework of the causal perturbation theory and address the problem of dynamical mass generation for the gauge boson. In this context we obtain an unambiguous expression for the coefficient of the induced Chern–Simons term.

1 Introduction

Recently there has been renewed interest in theories involving four-fermion interactions as a framework to the study of the top quark condensate [1]. In these models, based on the Nambu and Jona-Lasinio model [2], the top quark acquires mass when the four-fermion coupling constant is larger than a certain critical value. It has been shown that a similar behavior occurs in d- $(2 \leq d < 4)$ dimensional Thirring-like four-fermion interactions [3–7].

The Thirring model [8] was originally proposed as a soluble model for the interaction of fermions in $(1+1)$ dimensions. This original version had no local gauge symmetry. Since then, several authors [3–5], using a linearized version of the model by introducing an auxiliary vector field, have studied the Thirring model in d dimensions, in the context of $1/N$ expansion, taking into account a 'gauge fixing' term. However, as shown in [6,7,9], one can implement gauge invariance by using the Stückelberg formalism, so that the Thirring model emerges as a gauge-fixed version of a gauge theory. This model has been used to study fermion dynamical mass generation [6,7,10].

In $(2+1)$ dimensions these models exhibit a richer structure. Namely, for odd number of massive two-component fermions (the fermion masses may be present in the original Lagrangian or have dynamical origin) a Chern–Simons parity-breaking term is induced. This term is relevant to the quantum Hall effect [11] and, in particular, to dynamical mass generation for the associated vector field.

In this paper, we will consider the three-dimensional Thirring model as a gauge theory for one-fermion flavor. The approach we will adopt to study this model was proposed by Epstein and Glaser [12] in the 1970s and later applied to quantum electrodynamics (QED) by Scharf [13]. Their method, in which causality plays a central role, has the advantage that all physical quantities are mathematically well defined and ultraviolet divergences do not occur if one carefully carries out the splitting of distributions in the perturbation series. In particular, this implies that no ultraviolet cut-off has to be introduced and, for a nonrenormalizable theory, this is the interesting point.

Our main interest here is the coefficient of the induced Chern–Simons term, because it is generally stated that this coefficient is dependent on the regularization scheme used to treat the divergences [7,14], even though this difficulty has already been overcomed in usual treatments of QED_3 [15,16]. As shown in [17], the causal approach affords an unambiguous method for dealing with such problems.

This paper is organized as follows. In Sect. 2 we introduce the gauged version of the Thirring model [6,7], for one-fermion flavor, using the Stückelberg formalism. In Sect. 3 we give a brief presentation of the method of Epstein and Glaser. A proof of the nonrenormalizability of the Thirring model in the context of the causal method is given in Sect. 4. In Sect. 5 we obtain the vacuum polarization tensor and the modified gauge boson propagator, showing that there is generation of dynamical mass for the gauge boson as a function of the coupling constant. Section 6 is devoted to conclusions.

2 Thirring model as a gauge theory

Following closely [6] and [7], we present in this section the Thirring model as a gauge theory. In (2+1) dimensions the Lagrangian density for the massive Thirring model with one-fermion flavor is

$$\mathcal{L} = \overline{\psi}\mathrm{i}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi - \frac{G}{2}(\overline{\psi}\gamma^\mu\psi)(\overline{\psi}\gamma_\mu\psi), \tag{1}$$

where ψ is a two-component fermion field with mass m, supposed to be positive. The coupling constant G has dimension of $(\text{mass})^{-1}$ and will be redefined as $G = e^2/M^2$, with e a dimensionless parameter [18].

The algebra for the γ matrices in $(2+1)$ dimensions is realized using the Pauli matrices

$$\gamma^0 = \sigma_3, \quad \gamma^1 = \mathrm{i}\sigma_1, \quad \gamma^2 = \mathrm{i}\sigma_2, \tag{2}$$

with

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \gamma^\mu\gamma^\nu = g^{\mu\nu} - \mathrm{i}\varepsilon^{\mu\nu\delta}\gamma_\delta, \tag{3}$$

where $g_{\mu\nu} = \mathrm{diag}(1,-1,-1)$ and $\varepsilon^{\mu\nu\delta}$ is the totally antisymmetric Levi–Civita tensor.

We can linearize the four-fermion interaction by the introduction of an auxiliary vector field $\tilde{A}_\mu$, so that the Lagrangian is rewritten as

$$\mathcal{L}' = \overline{\psi}\mathrm{i}\gamma^\mu \tilde{D}_\mu\psi - m\overline{\psi}\psi + \frac{M^2}{2}\tilde{A}^\mu\tilde{A}_\mu, \tag{4}$$

with $\tilde{D}_\mu = \partial_\mu - \mathrm{i}e\tilde{A}_\mu$. It is important to note that, in spite of the formal similarity, $\tilde{D}_\mu$ is not a covariant derivative, because the Lagrangian density (4) does not have local gauge symmetry. The three-vector $\tilde{A}_\mu \equiv -\frac{e}{M^2}\overline{\psi}\gamma_\mu\psi$ is just a suitable representation of the current.

However, one can introduce a local gauge symmetry [9] making use of the Stückelberg formalism. Namely, we decompose the vector field $\tilde{A}_\mu$ according to $\tilde{A}_\mu = A_\mu - \partial_\mu\theta$, where A_μ is a vector field and θ a neutral scalar field, whereas we perform the change $\psi \to \mathrm{e}^{-\mathrm{i}e\theta}\psi$, $\overline{\psi} \to \overline{\psi}\mathrm{e}^{\mathrm{i}e\theta}$ (for a review of Stückelberg's formalism see [19]). Thus we get

$$\mathcal{L}'' = \overline{\psi}\mathrm{i}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi + \frac{M^2}{2}(A_\mu - \partial_\mu\theta)^2, \tag{5}$$

with $D_\mu = \partial_\mu - \mathrm{i}eA_\mu$. This Lagrangian is invariant under the gauge transformation

$$\begin{aligned} A_\mu &\to A'_\mu = A_\mu + \partial_\mu\phi, \\ \theta &\to \theta' = \theta + \phi, \\ \psi &\to \psi' = \mathrm{e}^{\mathrm{i}e\phi}\psi, \\ \overline{\psi} &\to \overline{\psi}' = \overline{\psi}\mathrm{e}^{-\mathrm{i}e\phi} \end{aligned} \tag{6}$$

so that A_μ is really a gauge field and D_μ a covariant derivative. From (6) we can see that in the unitary gauge $\theta' = 0$ one recovers the Lagrangian (1), i.e., the original Thirring model is just a gauge-fixed version of (5).

Since the Lagrangian (5) has local gauge symmetry, by adding a gauge-fixing and a Faddeev–Popov ghost term $\mathcal{L}_{GF+FP}$, we can obtain the complete BRST invariant Lagrangian [7]

$$\mathcal{L}_{Th,G} = \mathcal{L}'' + \mathcal{L}_{GF+FP}, \tag{7}$$

where $\mathcal{L}_{GF+FP}$ can be chosen in the form

$$\mathcal{L}_{GF+FP} = -\mathrm{i}\delta_B\left[\overline{c}\left(F[A,\theta] + \frac{\xi}{2}B\right)\right], \tag{8}$$

so that (7) is invariant under the BRST transformation

$$\begin{aligned} \delta_B\psi(x) &= iec(x)\psi(x), \\ \delta_B\theta(x) &= c(x), \\ \delta_B A_\mu(x) &= \partial_\mu c(x), \\ \delta_B\overline{c}(x) &= iB(x), \\ \delta_B c(x) &= 0, \\ \delta_B B(x) &= 0. \end{aligned} \tag{9}$$

In the above expressions δ_B represents the nilpotent BRST transformation, $c(x)$ and $\overline{c}(x)$ are the Faddeev–Popov ghosts and $B(x)$ is the Nakanishi–Lautrup auxiliary field.

When the functional $F[A,\theta]$ is linear in both A_μ and θ, the ghost fields decouple from the matter fields and, in particular, choosing the R_ξ gauge $F[A,\theta] = \partial_\mu A^\mu + \xi M^2\theta$, the Stückelberg field also decouples. So the Lagrangian (7), after integration over B, takes the form [6,7]

$$\mathcal{L}_{Th,G} = \mathcal{L}_{A,\psi} + \mathcal{L}_\theta + \mathcal{L}_{gh}, \tag{10}$$

where

$$\begin{aligned} \mathcal{L}_{A,\psi} &= \overline{\psi}\mathrm{i}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi + \frac{M^2}{2}A_\mu A^\mu \\ &\quad - \frac{1}{2\xi}(\partial_\mu A^\mu)^2, \\ \mathcal{L}_\theta &= \frac{1}{2}(\partial_\mu\theta)^2 - \frac{\xi M^2}{2}\theta^2, \\ \mathcal{L}_{gh} &= \mathrm{i}\left[(\partial_\mu\overline{c})(\partial^\mu c) - \xi M^2\overline{c}c\right]. \end{aligned} \tag{11}$$

Note that A_μ is not a dynamical field at tree level because there is no associated kinetic term in (11). Nevertheless, as we are going to show, the gauge boson acquires dynamics by radiative corrections. Another point which must be stressed is that, in the limit $\xi \to \infty$, we recover the original Thirring model.

As pointed in [6], the fact that the Lagrangian (11) has a gauge symmetry restricts the choice of the regularization schemes to be used, i.e., we only can employ the regularization schemes which preserve the gauge symmetry. This is the merit of the above construction in comparison with the naive use of the Lagrangian (4) with a gauge fixing term, without the prior introduction of a gauge symmetry (see [3,4]). Nevertheless, this is not sufficient to remove the regularization ambiguity in the coefficient of the induced Chern–Simons term when we calculate the fermion loop corrections. In this sense the causal method has been proven to be useful [13,17], because it never runs into the usual difficulties associated with ultraviolet divergences, resulting in a unambiguous value for the coefficient of the induced Chern–Simons term.

3 Epstein and Glaser theory

In the causal approach to quantum field theory, the S-matrix is viewed as an operator-valued distribution, written as

$$\begin{aligned} S(g) = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\int \mathrm{d}x_1\ldots\mathrm{d}x_n\, T_n(x_1,\ldots,x_n) \\ \times g(x_1)\ldots g(x_n). \end{aligned} \tag{12}$$

Once $T_1(x)$ is known, the S-matrix is constructed inductively, order by order, from the causal structure. The c-number test functions in (12) are supposed to belong to the rapidly decreasing Schwartz space functions, $g(x) \in \mathcal{S}(\mathbf{R}^3)$. The adiabatic limit $g \to 1$ must be considered at the end of the calculations.

Analogously, the inverse S-matrix has the form

$$S(g)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \tilde{T}_n(x_1, \ldots, x_n) \times g(x_1) \ldots g(x_n), \tag{13}$$

where $\tilde{T}_n(x)$ can be obtained by formal inversion of (12). Since the n-point function T_n ($\tilde{T}_n$) is symmetrical in its arguments we will use the notation $X = \{x_1, ..., x_n\}$.

The inductive step is as follows: if all $T_m(X)$, $m \leq n-1$ are known, one can define the distributions

$$A'_n(x_1, \ldots, x_n) = \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n),$$
$$R'_n(x_1, \ldots, x_n) = \sum_{P_2} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X), \tag{14}$$

where the sums run over all partitions

$$P_2 : \{x_1, \ldots, x_{n-1}\} = X \cup Y \ , \quad X \neq \emptyset, \tag{15}$$

into disjoint subsets with $|X| = n_1$, $|Y| \leq n-2$. If the sums are extended in order to include the empty set $X = \emptyset$ we get

$$A_n(x_1, \ldots, x_n) = \sum_{P_2^0} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n) = A'_n(x_1, \ldots, x_n) + T_n(x_1, \ldots, x_n), \tag{16}$$
$$R_n(x_1, \ldots, x_n) = \sum_{P_2^0} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X) = R'_n(x_1, \ldots, x_n) + T_n(x_1, \ldots, x_n),$$

where P_2^0 stands for all partitions

$$P_2^0 : \{x_1, \ldots, x_{n-1}\} = X \cup Y. \tag{17}$$

We can see that A_n and R_n in (16) are not known from the hypothesis of induction because they contain the unknown T_n. Only the difference

$$D_n(x_1, \ldots, x_n) = R'_n - A'_n = R_n - A_n, \tag{18}$$

is known.

If we use causality, it turns out that R_n has retarded support and A_n has advanced support, i.e.

$$\mathrm{supp} R_n(X) \subseteq \Gamma^+_{n-1}(x_n),$$
$$\mathrm{supp} A_n(X) \subseteq \Gamma^-_{n-1}(x_n), \tag{19}$$

with

$$\Gamma^{\pm}_{n-1}(x) \equiv \{(x_1, \ldots, x_{n-1}) \mid x_j \in \overline{V}^{\pm}(x), \forall j = 1, \ldots, n-1\}, \tag{20}$$
$$\overline{V}^{\pm}(x) = \{y \mid (y-x)^2 \geq 0, \pm(y^0 - x^0) \geq 0\}.$$

The distribution D_n has causal support, $\mathrm{supp} D_n \subseteq \Gamma^+_{n-1} \cup \Gamma^-_{n-1}$ and by decomposing D_n in advanced and retarded distributions we obtain the T_n distribution using (16).

The operator-valued distributions which we shall have to split are of the form

$$D_n(x_1, ..., x_n) = \sum_k : \prod_j \overline{\psi}(x_j) d_n^k(x_1, \ldots, x_n) \times \prod_l \psi(x_l) \prod_m A(x_m) :, \tag{21}$$

where ψ, $\overline{\psi}$ are the free fermion fields and A the free gauge boson fields. In this expression d_n^k are numerical tempered distributions, $d_n^k \in \mathcal{S}'(\mathbf{R}^{3n})$, with causal support. Because of the translation invariance, it is sufficient to put $x_n = 0$ and consider

$$d(x) \equiv d_n^k(x_1, \ldots, x_{n-1}, 0) \in \mathcal{S}'(\mathbf{R}^m), \quad m = 3n - 3. \tag{22}$$

The nontrivial step is the splitting of the numerical causal distribution d in the advanced and retarded distributions a and r, respectively. From the fact that $\Gamma^+(0) \cap \Gamma^-(0) = \{0\}$ we can see that the behavior of $d(x)$ in $x = 0$ is crucial in the splitting problem. For this reason, it is necessary to classify the singular distributions. With this aim we introduce the following definitions [13,20]:

Definition 1. The distribution $d(x) \in \mathcal{S}'(\mathbf{R}^m)$ has a quasi-asymptotics $d_0(x)$ at $x = 0$ with respect to a positive continuous function $\rho(\delta)$, $\delta > 0$, if the limit

$$\lim_{\delta \to 0} \rho(\delta) \delta^m d(\delta x) = d_0(x) \not\equiv 0, \tag{23}$$

exists in $\mathcal{S}'(\mathbf{R}^m)$.

The equivalent definition in momentum space reads

Definition 2. The distribution $\hat{d}(p) \in \mathcal{S}'(\mathbf{R}^m)$ has a quasi-asymptotics $\hat{d}_0(p)$ at $p = \infty$ if the limit

$$\lim_{\delta \to 0} \rho(\delta) \langle \hat{d}(\frac{p}{\delta}), \overset{\vee}{\phi}(p) \rangle = \langle \hat{d}_0, \overset{\vee}{\phi} \rangle, \tag{24}$$

exists for all $\overset{\vee}{\phi} \in \mathcal{S}(\mathbf{R}^m)$.

In (24) $\hat{d}$ denotes the distributional Fourier transform of d and $\overset{\vee}{\phi}$ the inverse Fourier transform of ϕ. The function $\rho(\delta)$ is called the power-counting function.

Definition 3. The distribution $d \in \mathcal{S}'(\mathbf{R}^m)$ is called singular of order ω if it has a quasi-asymptotics $d_0(x)$ at $x = 0$, or its Fourier transform has a quasi-asymptotics

$\hat{d}_0(p)$ at $p=\infty$, respectively, with power-counting function $\rho(\delta)$ satisfying

$$\lim_{\delta\to 0}\frac{\rho(c\delta)}{\rho(\delta)}=c^{\omega}, \tag{25}$$

for each $c>0$.

It follows [12,13] that for $\omega<0$ the solution is unique and can be defined by multiplication by a step function. For $\omega\geq 0$ the retarded distribution obtained from the causal splitting can be written down by means of the 'dispersion' formula [13]

$$\hat{r}(p)=\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}t\frac{\hat{d}(tp)}{(t-\mathrm{i}0)^{\omega+1}(1-t+\mathrm{i}0)}. \tag{26}$$

But, in contrast with the case $\omega<0$, this solution is not unique. If $\tilde{r}(x)$ is the retarded part of another decomposition, then $\tilde{r}(x)-r(x)$ is a distribution with support in $\{0\}$. In momentum space this gives the general solution of the splitting problem

$$\tilde{r}(p)=\hat{r}(p)+\sum_{|a|=0}^{\omega}C_a p^a, \tag{27}$$

where the constant coefficients C_a are not fixed by the causal structure; additional physical conditions are needed to determine them.

It is worth making some comments about the solutions (26) and (27). First, the solution (26), called the central splitting solution, preserves the symmetries of the theory, in special Lorentz covariance and gauge invariance. Second, in expression (27) the *minimal distribution splitting* condition, which says that the singular order cannot be raised in the splitting, was assumed. This condition, very important in QED_4 [13] and QED_3 [17,21], will also be useful here. Finally, the right singular order ω in (26) is essential, since if we underestimate ω, the integral in (26) will not be convergent and, again, one runs into the ultraviolet divergences of the usual perturbation theory.

4 Nonrenormalizability proof

The Lagrangian $\mathcal{L}_{A,\psi}$, (11), is our starting point for the causal treatment of the Thirring model as a gauge theory. From (11) we see that the first-order term in the causal perturbative expansion of the S-matrix is

$$T_1(x)=-\mathrm{i}e:\overline{\psi}(x)\gamma^{\mu}\psi(x):A_{\mu}=-\tilde{T}_1(x). \tag{28}$$

From this expression we see that the dimensionless parameter e plays the role of an expansion parameter, analogous to the electric charge in QED. But in the limit $\xi\to\infty$, when we recover the relation between $\tilde{A}_{\mu}$ and the fermion current in the original Thirring model, the true expansion parameter is G.

As pointed out in the last section, the A_{μ} field is not a dynamical field (the genuine dynamical field is $\partial_{\mu}A^{\mu}$). Therefore, there is no propagation associated with A_{μ}. Nevertheless, we can, formally, associate to A_{μ} a 'propagator'. From (11) we obtain the Feynman Green's function

$$D^{\mathrm{F}}_{\mu\nu}(k)=\frac{\mathrm{i}}{\sqrt{2\pi}}\frac{1}{M^2}\left(g_{\mu\nu}-\frac{k_{\mu}k_{\nu}}{k^2-\xi M^2}\right). \tag{29}$$

In the same way we get the commutation functions

$$D^{(\pm)}_{\mu\nu}(x)=\pm\frac{\mathrm{i}}{(2\pi)^2}\int\mathrm{d}^3k\,\frac{k_{\mu}k_{\nu}}{M^2}\delta(k^2-\xi M^2)\times\theta(\pm k_0)\mathrm{e}^{-\mathrm{i}k\cdot x}. \tag{30}$$

This enables us to obtain the second-order distributions T_2. However, before starting the perturbation theory, it is useful to derive a general expression for the singular order ω of arbitrary graphs.

Proposition 1. For the Thirring model the singular order is

$$\omega=3-f-\frac{3}{2}b+\frac{1}{2}n, \tag{31}$$

where f (b) is the number of external fermions (bosons) and n is the order of perturbation theory.

Proof. The proof is by induction [13,17]. First we verify (31) for the diagrams in lowest order. The first-order term (28) has $\omega=0$, by definition.

Then, to verify that this relation is preserved in the step from $n-1$ to n in perturbation theory, we must consider a tensor product of two subgraphs with singular order ω_1 and ω_2 which satisfies (31), by hypothesis. This tensor product has to be normally ordered, giving rise to bosonic and fermionic contractions. Here we will consider only the bosonic case, since fermionic contractions have already been considered in the context of QED_3 [17,21].

Suppose that l bosonic contractions arise in the process. Then the numerical distribution of the contracted expression is

$$\begin{aligned}&t_1^{[\mu]}(x_1-x_r,\dots,x_{r-1}-x_r)\prod_{j=1}^{l}D^{(+)}_{\mu_j\nu_j}(x_{r_j}-y_{v_j})t_2^{[\nu]}\\&\times(y_1-y_v,\dots,y_{v-1}-y_v)\\&\quad\equiv t(\zeta_1,\dots,\zeta_{r-1},\eta_1,\dots,\eta_{v-1},\eta),\end{aligned} \tag{32}$$

where we have taken into account the translation invariance. In this expression $\{x_{r_j}\}$ is a subset of $\{x_1,\dots,x_r\}$ and $\{y_{v_j}\}$ is a subset of $\{y_1,\dots,y_v\}$. We have introduced the relative coordinates

$$\zeta_j=x_j-x_r,\quad \eta_j=y_j-y_v,\quad \eta=x_r-y_v, \tag{33}$$

and the superscripts $[\mu]$ and $[\nu]$ mean the collection of indices $\{\mu_1,\dots,\mu_l\}$ and $\{\nu_1,\dots,\nu_l\}$, respectively.

The Fourier transform of $t(\zeta,\eta)$ in (32), taking into account the fact that products go into convolutions, is

$$\begin{aligned}&\hat{t}(p_1,\dots,p_{r-1},q_1,\dots,q_{v-1},q)\\&\quad\propto\int\left(\prod_{j=1}^{l}\mathrm{d}^3k_j\right)\delta^{(3)}(q-\sum_{j=1}^{l}k_j)\hat{t}_1^{[\mu]}(\dots,p_i-k_{r(i)},\dots)\end{aligned}$$

$$\times \prod_{j=1}^{l} \hat{D}^{(+)}_{\mu_j\nu_j}(k_j)\hat{t}_2^{[\nu]}(\ldots, q_s + k_{v(s)}, \ldots), \quad (34)$$

where $r(i) = v(s)$ if and only if x_i and y_s are joined by a contraction. For the coordinates x_j and y_m which are not joined by a contraction we have just p_j and q_m as arguments, respectively. The proportionality sign is to indicate that we are omitting powers of 2π.

Applying $\hat{t}(p_1, \ldots, q)$ in a test function $\overset{\vee}{\phi} \in \mathcal{S}(\mathrm{R}^{3(r+v-1)})$ we get, after some algebra,

$$\langle \hat{t}, \overset{\vee}{\phi} \rangle \propto \int \mathrm{d}^{3r-1}p'\, \mathrm{d}^{3v}q'\, \hat{t}_1^{[\mu]}(p')\hat{t}_2^{[\nu]}(q')\psi_{[\mu\nu]}(p', q'), \quad (35)$$

where $\psi_{[\mu\nu]}(p', q')$ is defined as

$$\psi_{[\mu\nu]}(p', q') = \int \left(\prod_{j=1}^{l} \mathrm{d}^3k_j\right) \mathrm{d}^3q'\, \delta^{(3)} \times (q' - \sum_{j=1}^{l} k_j) \prod_{j=1}^{l} \hat{D}^{(+)}_{\mu_j\nu_j}(k_j) \times \overset{\vee}{\phi}(\ldots, p_i' + k_{r(i)}, \ldots, q_s' - k_{v(s)}, \ldots, q'). \quad (36)$$

In order to determine the singular order of $\hat{t}$, according to definition 2, we have to consider the scaled distribution $\hat{t}(\frac{p_1}{\delta}, \ldots, \frac{q}{\delta})$. Then, we find

$$\langle \hat{t}(\frac{p_1}{\delta}, \ldots, \frac{q}{\delta}), \overset{\vee}{\phi} \rangle = \delta^m \int \mathrm{d}^{3r-3}p'\, \mathrm{d}^{3v-3}q'\, \hat{t}_1^{[\mu]}(p')\hat{t}_2^{[\nu]}(q') \times \psi_{\delta[\mu\nu]}(p', q'), \quad (37)$$

with $m = 3(r + v - 1)$ and

$$\psi_{\delta[\mu\nu]}(p', q') = \int \left(\prod_{j=1}^{l} \mathrm{d}^3k_j\right) \mathrm{d}^3q'\, \delta^{(3)} \times \left(q' - \sum_{j=1}^{l} k_j\right) \prod_{j=1}^{l} \hat{D}^{(+)}_{\mu_j\nu_j}(k_j) \times \overset{\vee}{\phi}(\ldots, \delta(p_i' + k_{r(i)}), \ldots, \delta(q_s' - k_{v(s)}), \ldots, \delta q'). \quad (38)$$

We introduce the new variables $\tilde{k}_j = \delta k_j$ and $\tilde{q} = \delta q$, and observing that

$$\hat{D}^{(+)}_{\mu\nu}(\frac{\tilde{k}}{\delta}) = \frac{\tilde{k}_\mu\tilde{k}_\nu}{\delta^2 M^2}\delta^{(1)}\left(\frac{\tilde{k}^2}{\delta^2} - \xi M^2\right)\theta\left(\frac{\tilde{k}_0}{\delta}\right) = \frac{\tilde{k}_\mu\tilde{k}_\nu}{M^2}\delta^{(1)}(\tilde{k}^2)\theta(k_0) \equiv \hat{D}^{(+)}_{0\mu\nu}(\tilde{k}), \quad (39)$$

we obtain

$$\psi_{\delta[\mu\nu]}(p, q) = \frac{1}{\delta^{3l}}\psi^0_{[\mu\nu]}(\delta p, \delta q), \quad (40)$$

where the superscript 0 indicates that $\hat{D}^{(+)}_{\mu\nu}$ is replaced by $\hat{D}^{(+)}_{0\mu\nu}$ in (36). Then, using this result and $\delta p = \tilde{p}$, $\delta q = \tilde{q}$, we find from (37)

$$\left\langle \hat{t}\left(\frac{p_1}{\delta}, \ldots, \frac{q}{\delta}\right), \overset{\vee}{\phi} \right\rangle = \delta^{3-3l} \int \mathrm{d}^{3r-3}\tilde{p}\, \mathrm{d}^{3v-3}\tilde{q}\, \hat{t}_1^{[\mu]} \times \left(\frac{\tilde{p}}{\delta}\right) \hat{t}_2^{[\nu]} \left(\frac{\tilde{q}}{\delta}\right) \psi^0_{[\mu\nu]}(\tilde{p}, \tilde{q}). \quad (41)$$

But, by the induction hypothesis, the distributions $\hat{t}_1^{[\mu]}$ and $\hat{t}_2^{[\nu]}$ have singular orders ω_1 and ω_2 with power-counting functions $\rho_1(\delta)$ and $\rho_2(\delta)$, respectively. So we verify that the limit considered in definition 2 exists for the distribution $\hat{t}$ with power-counting function given by

$$\rho(\delta) = \delta^{3l-3}\rho_1(\delta)\rho_2(\delta), \quad (42)$$

with singular order

$$\omega = 3l - 3 + \omega_1 + \omega_2. \quad (43)$$

Thus, substituting

$$\omega_i = 3 - f_i - \frac{3}{2}b_i + \frac{1}{2}n_i, \quad (44)$$

for ω_1 and ω_2 gives

$$\omega = 3 - (f_1 + f_2) - \frac{3}{2}(b_1 + b_2 - 2l) + \frac{1}{2}(n_1 + n_2), \quad (45)$$

which proves the above proposition for l bosonic contractions.

From (31) we have that, for one-loop corrections, the vacuum polarization ($n = 2$, $f = 0$, $b = 2$) has $\omega_{\mathrm{vp}} = 1$, the fermion self-energy ($n = 2$, $f = 2$, $b = 0$) has $\omega_{\mathrm{se}} = 2$ and the vertex correction ($n = 3$, $f = 2$, $b = 1$), $\omega_v = 1$. We consider here only the vacuum polarization tensor. The fermion self-energy and the vertex correction will be considered elsewhere [22].

In addition, from proposition 1 it follows that the Thirring model is a nonrenormalizable theory, as expected. This means that the number of free parameters, i.e. the coefficients of the polynomial in p in (27), increases indefinitely when we consider higher orders in perturbation theory such that we cannot fix all of them by symmetry considerations. However, as we shall see in the next section, for the vacuum polarization tensor we will be able to determine all constants appearing in second-order perturbation theory.

5 Dynamical mass generation

In this section we consider the vacuum polarization and address the dynamical generation of a kinetic term for the gauge boson. Since the vacuum polarization tensor assumes the same form as that in QED_3, we omit the details of the calculation, which can be found in [13,17].

In second-order perturbation theory we can construct the distribution $D_2(x_1, x_2) = R_2' - A_2'$ following the steps outlined in Sect. 1. So, using Wick's theorem, the contribution for the vacuum polarization in D_2 is

$$D_{2\text{vp}}(x_1,x_2) = -e^2\text{Tr}\left[\gamma^\mu S^{(-)}(y)\gamma^\nu S^{(+)}(-y) -\gamma^\mu S^{(+)}(y)\gamma^\nu S^{(-)}(-y)\right] \times :A_\mu(x_1)A_\nu(x_2):, \quad (46)$$

where $y \equiv x_1 - x_2$ and

$$S^{(\pm)}(x) = \pm\frac{\text{i}}{(2\pi)^2}\int \text{d}^3p\,(\not p + m)\theta(\pm p_0) \times\delta(p^2-m^2)\text{e}^{-\text{i}p\cdot x}. \quad (47)$$

The numerical distribution associated with $D_2(x_1,x_2)$ can be written in the form

$$d^{\mu\nu}(x_1,x_2) = P^{\mu\nu}(y) - P^{\nu\mu}(-y), \quad (48)$$

where

$$P^{\mu\nu}(y) \equiv e^2\text{Tr}\left[\gamma^\mu S^{(+)}(y)\gamma^\nu S^{(-)}(-y)\right]. \quad (49)$$

The distribution $d^{\mu\nu}(y)$ can be shown to have causal support [13,17]. So we can proceed the splitting according to the procedure already explained, following closely [17]. In momentum space, using (47) for the fermion commutation functions, $P^{\mu\nu}(k)$ can be written as

$$\hat{P}^{\mu\nu}(k) = -\frac{e^2}{(2\pi)^{\frac{5}{2}}}\int \text{d}^3p\,\theta(p_0)\delta(p^2-m^2)\theta(k_0-p_0) \times\delta[(k-p)^2-m^2]j^{\mu\nu}(k,p), \quad (50)$$

with

$$\begin{aligned} j^{\mu\nu}(k,p) &= \text{Tr}\left[\gamma^\mu(\not p + m)\gamma^\nu(\not k - \not p - m)\right] \\ &= -2[(m^2-p^2)g^{\mu\nu} + 2p^\mu p^\nu - (p^\mu k^\nu + k^\mu p^\nu) \\ &\quad + g^{\mu\nu}p\cdot k + \text{i}m\varepsilon^{\mu\nu\delta}k_\delta]. \end{aligned} \quad (51)$$

From (50) and (51) one can observe that $P^{\mu\nu}$ is gauge invariant

$$k_\mu\hat{P}^{\mu\nu}(k) = 0. \quad (52)$$

This property enables us to attribute to $P^{\mu\nu}$ the following tensor structure

$$\hat{P}^{\mu\nu}(k) = \hat{P}^{\mu\nu}_s(k) + \hat{P}^{\mu\nu}_a(k), \quad (53)$$

with

$$\hat{P}^{\mu\nu}_s(k) = (k^\mu k^\nu - k^2g^{\mu\nu})\tilde{B}_1(k^2), \quad (54)$$

$$\hat{P}^{\mu\nu}_a(k) = \text{Im}\varepsilon^{\mu\nu\delta}k_\delta\tilde{B}_2(k^2). \quad (55)$$

Projecting $\tilde{B}_1(k^2)$ and $\tilde{B}_2(k^2)$ from $\hat{P}^{\mu\nu}(k)$, one obtains

$$\tilde{B}_1(k^2) = -\frac{e^2}{(2\pi)^{\frac{3}{2}}}\frac{1}{8\sqrt{k^2}}\left(1+\frac{4m^2}{k^2}\right) \times\theta(k_0)\theta(k^2-4m^2), \quad (56)$$

and

$$\tilde{B}_2(k^2) = \frac{e^2}{(2\pi)^{\frac{3}{2}}}\frac{1}{2\sqrt{k^2}}\theta(k_0)\theta(k^2-4m^2). \quad (57)$$

Since the Fourier transform of $P^{\mu\nu}(-y)$ is given by $\hat{P}^{\mu\nu}(-k)$, from (48) we see that

$$\hat{d}^{\mu\nu}(k) = \hat{P}^{\mu\nu}(k) - \hat{P}^{\nu\mu}(-k) = \hat{d}^{\mu\nu}_s(k) + \hat{d}^{\mu\nu}_a(k), \quad (58)$$

where

$$\hat{d}^{\mu\nu}_s(k) = (k^\mu k^\nu - k^2g^{\mu\nu})B_1(k^2), \quad (59)$$

$$\hat{d}^{\mu\nu}_a(k) = \text{i}m\varepsilon^{\mu\nu\delta}k_\delta B_2(k^2), \quad (60)$$

with $B_1(k^2)$ and $B_2(k^2)$ given by expressions (56) and (57), replacing $\theta(k_0)$ by $\text{sgn}(k_0)$.

From (59) and (60) we find that the singular orders of $\hat{d}^{\mu\nu}_s(k)$ and $\hat{d}^{\mu\nu}_a(k)$ are $\omega_s = 1$ and $\omega_a = 0$, respectively. Then the distribution splitting is nontrivial ($\omega \geq 0$) and we need to use the central splitting solution, (26), with the appropriate ω to obtain the retarded distribution. Since $\hat{d}^{\mu\nu}_s$ and $\hat{d}^{\mu\nu}_a$ are independent due to the tensor structure, the splitting process for each one must be considered separately.

For the symmetric part we have

$$\hat{r}^{\mu\nu}_s(k) = \frac{\text{i}}{2\pi}(k^\mu k^\nu - k^2g^{\mu\nu}) \times\int_{-\infty}^{+\infty}\text{d}t\,\frac{t^2B_1(t^2k^2)}{(t-\text{i}0)^2(1-t+\text{i}0)}, \quad (61)$$

which results in

$$\hat{r}^{\mu\nu}_s(k) = \frac{\text{i}}{(2\pi)^{\frac{3}{2}}}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\Pi^{(1)}(k^2), \quad (62)$$

with

$$\Pi^{(1)}(k^2) = \frac{e^2}{16\pi}k^2\text{sgn}(k_0)\left[\frac{4m}{k^2} + \frac{1}{\sqrt{k^2}}\left(1+\frac{4m^2}{k^2}\right) \times\left(\ln\left|\frac{1-\sqrt{\frac{k^2}{4m^2}}}{1+\sqrt{\frac{k^2}{4m^2}}}\right| - \text{i}\pi\theta(k^2-4m^2)\right)\right]. \quad (63)$$

Since the singular order of $\hat{d}^{\mu\nu}_s$ is $\omega_s = 1$, from (27) we have for the general solution of the splitting

$$\tilde{r}^{\mu\nu}_s(k) = \hat{r}^{\mu\nu}_s(k) + C_0g^{\mu\nu} + C'_\delta\varepsilon^{\mu\nu\delta} + C^\mu_1k^\nu + C^\nu_2k^\mu, \quad (64)$$

where C_0, C'_δ, C^μ_1 and C^ν_2 are constants which are not fixed by causality. However, $C'_\delta = 0$ to preserve the symmetric structure. The Lorentz structure and the fact that C^μ_1 and C^ν_2 are c-numbers lead to $C^\mu_1 = C^\nu_2 = 0$. By the requirement of gauge invariance, $k_\mu\tilde{r}^{\mu\nu}_s(k) = 0$, C_0 must vanish. So, the general solution of the splitting problem for the symmetrical part is given by (62).

For the antisymmetric part we use (26) with $\omega = 0$

$$\hat{r}_a^{\mu\nu}(k) = -\frac{m}{2\pi}\varepsilon^{\mu\nu\delta}k_\delta \int_{-\infty}^{+\infty} \mathrm{d}t\, \frac{tB_2(t^2k^2)}{(t-\mathrm{i}0)(1-t+\mathrm{i}0)}, \quad (65)$$

from which we obtain

$$\hat{r}_a^{\mu\nu}(k) = -\frac{m}{(2\pi)^{\frac{3}{2}}}\varepsilon^{\mu\nu\delta}k_\delta \Pi^{(2)}(k^2), \quad (66)$$

where

$$\Pi^{(2)}(k^2) = \frac{e^2}{4\pi}\frac{\mathrm{sgn}(k_0)}{\sqrt{k^2}} \times \left[\ln\left|\frac{1-\sqrt{\frac{k^2}{4m^2}}}{1+\sqrt{\frac{k^2}{4m^2}}}\right| - \mathrm{i}\pi\theta(k^2-4m^2)\right]. \quad (67)$$

The general solution for the antisymmetric part is

$$\tilde{r}_a^{\mu\nu}(k) = \hat{r}_a^{\mu\nu}(k) + C_0 g^{\mu\nu} + C_{1\delta}\varepsilon^{\mu\nu\delta}. \quad (68)$$

However, the constant C_0 must vanish to preserve the antisymmetric structure and $C_{1\delta}$ must also vanish by gauge invariance, so that the general solution of the splitting of the antisymmetric part is given by (66).

At this point, it is interesting to note that, in spite of the fact that the model is nonrenormalizable, we were able to determine all constants C_a appearing in the general solution for the polarization tensor. Of course, this is not always the case. In fact, for the fermion self-energy and the vertex correction there remains one undetermined constant [22].

The vacuum polarization tensor is defined as

$$\Pi_{\mu\nu}(k) = -\mathrm{i}(2\pi)^{\frac{3}{2}}\left(\hat{r}_{\mu\nu}(k) - \hat{r}'_{\mu\nu}(k)\right), \quad (69)$$

where $\hat{r}'_{\mu\nu}(k)$ is the Fourier transform of the numerical distribution associated with $R'_2(x_1,x_2)$, (14). In this case, $\hat{r}'_{\mu\nu}(k) = -\hat{P}_{\mu\nu}(-k)$ and, from (49), we see that this distribution do not contribute in the region $k^2 < 4m^2$. Thus we can write down the polarization tensor as

$$\Pi^{\mu\nu}(k) = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\Pi^{(1)}(k^2) + \mathrm{i}m\varepsilon^{\mu\nu\delta}k_\delta \Pi^{(2)}(k^2), \quad (70)$$

with $\Pi^{(1)}(k^2)$ and $\Pi^{(2)}(k^2)$ given by (63) and (67), respectively, satisfying

$$\Pi^{(1)}(0) = 0, \qquad \Pi^{(2)}(0) = -\frac{e^2}{4\pi m}. \quad (71)$$

Let us now derive the gauge boson propagator modified by vacuum polarization insertions, in the one-loop approximation. This is given by the series

$$\mathcal{D} = D_F + \mathrm{i}D_F\Pi D_F + \mathrm{i}D_F\Pi D_F\Pi D_F + \ldots = D_F + \mathrm{i}D_F\Pi\mathcal{D}, \quad (72)$$

from which

$$\mathcal{D}_{\mu\nu}^{-1} = (D_{\mu\nu}^{\mathrm{F}})^{-1} - \mathrm{i}\Pi_{\mu\nu}, \quad (73)$$

where D_F is the free gauge boson propagator, (29).

Here, it is important to note that, by the Coleman–Hill theorem [23], this approximation for the gauge boson propagator gives the exact contribution for the topological mass term. In what follows we reproduce the Coleman–Hill argument in the context of distribution theory [24].

Let us consider an n-gauge boson 'effective vertex' given by the sum of all graphs consisting of a single closed fermion loop with n external gauge bosons attached. Associated to this vertex we have a numerical regular distribution $\hat{t}_{\mu_1\ldots\,\mu_n}(k_1,\ldots,k_n)$, a generalized function of the $n-1$ independent momenta. By convention, we will take the first $n-1$ momenta as the independent ones and k_n fixed by momentum conservation. We will consider the distribution $\hat{t}_{\mu_1\ldots\,\mu_n}$ in Euclidean space, where it is an analytic generalized function of the momenta. So, gauge invariance entails

$$k_1^{\mu_1}\hat{t}_{\mu_1\ldots\,\mu_n}(k_1,\ldots,k_n) = 0. \quad (74)$$

Differentiating with respect to k_1^ν and taking $k_1^\nu = 0$, we get

$$\hat{t}_{\mu_1\ldots\mu_n}(0,k_2,\ldots,k_n) = 0, \quad (75)$$

or, expanding in Taylor series (remember that $\hat{t}_{\mu_1\ldots\mu_n}$ is a regular distribution)

$$\hat{t}_{\mu_1\ldots\,\mu_n}(k_1,\ldots,k_n) = \mathcal{O}(k_1). \quad (76)$$

In the same way, $\hat{t}_{\mu_1\ldots\,\mu_n}$ is also $\mathcal{O}(k_2)$. Since, for $n > 2$, k_1 and k_2 are independent variables, we have

$$\hat{t}_{\mu_1\ldots\,\mu_n}(k_1,\ldots,k_n) = \mathcal{O}(k_1k_2), \qquad n > 2. \quad (77)$$

This shows that $\hat{t}_{\mu_1\ldots\,\mu_n}$ must be, at least, $\mathcal{O}(k_1\ldots k_{n-1})$. However, by Lorentz structure, it turns out that $\hat{t}_{\mu_1\ldots\,\mu_n}$ is $\mathcal{O}(k_1\ldots k_n)$ for $n > 2$ (see appendix of [23]).

Then, we can construct a gauge boson self-energy graph contracting bosonic lines of fermion loops (this implies contracting $\hat{t}_{\mu_1\ldots\,\mu_n}$ with the commutation function $D_{\mu\nu}^{(+)}(k)$, (30)). Contracting all lines of a graph but two, which are the external lines of the graph carrying momenta k and $-k$, we have three possibilities: (i) the two external lines are attached to distinct loops; (ii) the two external lines end at the same loop, but this has more than two bosonic lines; (iii) the two external lines end at the same loop and this has only two bosonic lines. In cases (i) and (ii) the corresponding distributions are $\mathcal{O}(k^2)$ due to (76) and (77), respectively. But from (70) and (73) we see that the topological mass is given by the coefficient of the term linear in k when $k^2 \to 0$. So, the only contribution to the topological mass comes from the second-order perturbation theory, case (iii) above.

One can see that gauge invariance plays a central role in the derivation of this result and, again, we see the relevance of the construction outlined in Sect. 1 [6,7]. In addition, we must observe that in this theory there are no infrared difficulties because the gauge bosons are massive.

Let us now turn to the inversion of (73). This is more easily performed by introducing the following projection operators

$$P^{\mu\nu}_{(1)} = \frac{1}{2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} + \mathrm{i}\varepsilon^{\mu\nu\delta}\frac{k_\delta}{\sqrt{k^2}}\right),$$
$$P^{\mu\nu}_{(2)} = \frac{1}{2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} - \mathrm{i}\varepsilon^{\mu\nu\delta}\frac{k_\delta}{\sqrt{k^2}}\right), \tag{78}$$
$$P^{\mu\nu}_{(3)} = \frac{k^\mu k^\nu}{k^2}.$$

This orthonormal set of operators satisfies the relation

$$\sum_{i=1}^{3} P^{\mu\nu}_{(i)} = g^{\mu\nu}. \tag{79}$$

Then, the free gauge boson propagator can be written as a linear combination of these projection operators as

$$D^{\mu\nu}_F = \frac{\mathrm{i}}{M^2}\left(P^{\mu\nu}_{(1)} + P^{\mu\nu}_{(2)} - \frac{\xi M^2}{k^2 - \xi M^2}P^{\mu\nu}_{(3)}\right), \tag{80}$$

such that the inversion is a trivial task. We just write down its inverse,

$$(D^{\mu\nu}_F)^{-1} = -\mathrm{i}M^2\left(P^{\mu\nu}_{(1)} + P^{\mu\nu}_{(2)} - \frac{k^2 - \xi M^2}{\xi M^2}P^{\mu\nu}_{(3)}\right). \tag{81}$$

In the same way we write the polarization tensor as

$$\Pi^{\mu\nu}(k) = (P^{\mu\nu}_{(1)} + P^{\mu\nu}_{(2)})\Pi^{(1)}(k^2) + m\sqrt{k^2}(P^{\mu\nu}_{(1)} - P^{\mu\nu}_{(2)})\Pi^{(2)}(k^2). \tag{82}$$

Introducing these expressions in (73) we get

$$\mathcal{D}^{\mu\nu} = -\frac{\mathrm{i}}{k^2 - \tilde{\Pi}(k^2)}\left[\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\frac{M^2 + \Pi^{(1)}(k^2)}{[m\Pi^{(2)}(k^2)]^2} - \mathrm{i}\varepsilon^{\mu\nu\delta}\frac{k_\delta}{m\Pi^{(2)}(k^2)}\right] - \mathrm{i}\xi\frac{k^\mu k^\nu}{k^2(k^2 - \xi M^2)}, \tag{83}$$

where

$$\tilde{\Pi}(k^2) = \frac{(M^2 + \Pi^{(1)}(k^2))^2}{[m\Pi^{(2)}(k^2)]^2}. \tag{84}$$

The form of the corrected propagator indicates that a pole is generated by the fermion loop insertions, that is, the gauge boson acquires a dynamical mass through the loop effects. Calling M_{gb} the mass of the gauge boson, we see that it is given by the solution of the transcendental equation

$$\left(mM_{\mathrm{gb}}\Pi^{(2)}(M^2_{\mathrm{gb}})\right)^2 = \left(M^2 + \Pi^{(1)}(M^2_{\mathrm{gb}})\right)^2, \quad \text{for } 0 \le M_{\mathrm{gb}} < 2m. \tag{85}$$

Before continuing the analysis of the equation above we should observe that the limit $G \to \infty$, which must be taken with e fixed, is well defined in a general gauge (although it is ill-defined in the unitary gauge, $\xi \to \infty$), as we can see from (29) and (83) [6,7]. Thus we verify that there is a solution of (85) for all G for $G > 0$. In particular, for $G \to \infty$ we get $M_{\mathrm{gb}} = 0$, as we can see by noticing that, in this limit, $M \to 0$ and, therefore,

$$\tilde{\Pi}(k^2)_{M^2=0} = \frac{(\Pi^{(1)}(k^2))^2}{[m\Pi^{(2)}(k^2)]^2}, \tag{86}$$

while, from (71), we see that $\tilde{\Pi}(0)_{M^2=0} = 0$, so that $M_{\mathrm{gb}} = 0$ is, in fact, a solution.

On the other side, for $G \to 0_+$ we have

$$\frac{M_{\mathrm{gb}}}{2m} = 1 - \alpha\exp\left(-\frac{2\pi}{mG}\right), \tag{87}$$

where $\alpha = 2e^{-\frac{1}{2}}$.

The solutions of the transcendental equation (85) in these two cases are consistent with the results expected. The A_μ field can be thought of as a bound state of two fermions. So, in the limit of very weak interaction, we expect that the mass of the vector channel is $2m$. On the other hand, in the limit of strong interaction we expect that the mass of A_μ goes to zero.

Finally, we should observe that the expression for the corrected propagator (83) has a well defined limit $m \to 0$, but in this case there is no pole for time-like momentum.

6 Conclusions

In this paper we have studied the massive gauged Thirring model in the context of the Epstein and Glaser's causal theory, and derived a proof of the nonrenormalizability of the model, obtaining a general expression for the singular order of the distributions associated with an arbitrary process.

In the sequence, we have obtained the vacuum polarization tensor by using the causal theory and have shown that the gauge boson, which at level three is an auxiliary field, becomes dynamical. The causal method naturally afforded the correct number of subtractions for the antisymmetric part of the vacuum polarization tensor and enabled us to determine the coefficient of the induced Chern–Simons term without ambiguity. Yet, the existence of a gauge symmetry leads to a result such that $e^2 \equiv GM^2$, according to the Coleman–Hill theorem.

We have also solved the transcendental equation for the pole of the gauge boson in the opposite cases of very strong and very weak coupling, obtaining results in accordance with those of [5,7].

Finally, it is important to note that in applying the causal method we never run into the usual ultraviolet divergences. Therefore, for nonrenormalizable models there is no necessity for a cut-off. However, even for renormalizable or super-renormalizable theories [13], we still have constants that are not fixed by causality. In the case of renormalizable and super-renormalizable theories these constants are determined by physical requirements, while for nonrenormalizable theories there remain a number of undetermined constants, which increase with the order of perturbation theory.

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